SEMINAR NOTES.

SOCIAL INSTITUTIONS AND THE RIEMANN SURFACE.[1]

WE are groping for a more perfect conception of the organic unity of the life of society. Such phrases as "the social organism" have for some time been frequent upon our lips, yet they do not carry, to the uninitiated at least, the whole significance of the light of a truth which is only beginning to dawn upon us. Attempted generalizations, where they have dealt with actual, known conditions, have been only partially successful. The vastness of the field for investigation, the complexity of the material to be dealt with, have rendered it extremely difficult to present results at once comprehensive and profound.

These facts have been realized by certain present-day philosophers whose first care it has been to elaborate and perfect a method which should be a tool worthy of the work. They have tried and tested it in many ways, and by means of it are laying the foundations of a philosophy which they believe shall be a vantage-ground to a better understanding of the nature of the structure of our social life. With the fruits of their labors our interest is here concerned. The purpose of the present endeavor is to lead to a realization of the significance of the new philosophy in one or two of its aspects. We do not ask how it has been derived, but what, in point of fact and formulæ, it is, as applied to all and every phase of life.

Instead of singling out a particular phenomenon, to trace its winding way through the tangle for a little distance, without being able to tell why it takes now this turn and now that, or to say anything about the myriads of other threads which cross and recross it—all that men have been able to do up to date—the sociologist today

[1] When the figure, JOURNAL OF SOCIOLOGY, November, 1898, p. 382, was explained to the class of which the writer of this paper was a member, Miss Hewes suggested that the thought could be more fully indicated by the symbolism of the Riemann doctrine. She was requested to elaborate the suggestion, and the paper may accordingly be read as an appendix to the chapter above cited. As its two closing paragraphs clearly indicate, it is not an attempt to give final formulation to social combinations. It tries to make the fact of the *complexity* in all social reactions more evident, and to give an approximate notion of the degree of that complexity. Miss Hewes' contribution to the subject is certainly commendable. A. W. S.

aims, from his point of view, *to think the whole human race* as a complex unity. He is striving to make more clear his perception of an entity, infinitely complex, in which he sees manifest certain causal relations, upon the further investigation of which he seeks, and believes it possible to discover, the underlying principles which operate over the whole plane of associated activity.

The forces that produce motion in social groups are the sum of the wants and desires of human beings. Let us accept the classification of them into (*a*) health, (*b*) wealth, (*c*) sociability, (*d*) knowledge, (*e*) beauty, (*f*) righteousness,[1] and signify them by the letters *a*, *b*, *c*, *d*, *e*, *f*. Now, these operate in varying proportion in every individual, and take different forms in different groups in various parts of the world during successive ages. But gradually, everywhere and always, the ways of gratifying them become customary and grow into corresponding social institutions. Thus, for example, it will readily be seen how from desires for wealth have grown up what were at first only conventional ways of gratification and have since become our various economic institutions, which are now obligatory ways of satisfying the wealth desire. (Of course — especially true in the example just cited — the direction is always primarily determined by the physical environment, but, considering the latter constant for the time given, we confine our attention rather to the psychic elements.) These institutions, then, must also be submitted to classification. We accept as the most comprehensive and inclusive grouping yet made De Greef's schedule of social phenomena.[2] They fall into seven principal groups: (*G*) economic, (*H*) genetic, (*I*) artistic, (*J*) scientific, (*K*) moral, (*L*) judicial, (*M*) political, to be designated respectively by the letters *G*, *H*, *I*, *J*, *K*, *L*, *M*.

"We have now for our problem [quoting Small] the discovery of the general laws of interrelation between the individual elements in society, represented in terms of desire by the product *a b c d e f*, and the institutional element, represented by *G H I J K L M* . . . . This discovery must be made by investigation of such reactions, both in selected eras, prehistoric, ancient, modern, or contemporary, and in successive civilizations, that is, it must be both static and dynamic. . . . This knowledge of the relation of individuals to institutions is a scientific desideratum;" and is to be attained only by scientific methods of investigation.

[1] See SMALL AND VINCENT, *Introduction to the Study of Society*, p. 175.

[2] See DE GREEF, *Introduction à la sociologie*, Vol. I, p. 214.

Observe what is affirmed in this hypothesis. Its significance lies in the question: Does there in actual fact exist between institutions and the constituent desires of which they are the product the relation of function to variable? And what, indeed, is the nature of that relation? Recall the definition of function given by Osborne. He says: "When the value of one variable quantity so depends upon that of another that any change in the latter produces a corresponding change in the former, the former is said to be a *function* of the latter." Thus

$$x^2, \quad \log(x^2+1), \quad \sqrt{x(x+1)}, \quad \text{etc.},$$

are functions of x. "A function of two or more variables is the relation existing between them such that any change in the one produces a corresponding change in the other." Thus

$$xy+yz+zx, \quad \sqrt{\frac{x+y}{z}},$$

are functions of x, y, and z. These illustrations symbolize the functional relation of social institutions and the human desires, which is no less intimate and fundamental than are the values of x, y, and z to the value of the expression.[1]

It may be that the human desires named do not embrace all the factors. (In point of fact, they do not.) There may enter in certain constants from the material environment which determine in part the form of the function and the paths of the variables (a, b, c, d, e, f) which we are trying to trace in order that we may have the means furnished us of ascertaining in any given case which of the n different values is properly assignable to any special function considered at a special point. For we find that the quantities with which we have to deal are *complex*, and our functions are *multiform*. Our analogy, then, to be understood, requires that we have in mind the method of representation of the imaginary quantities, and for convenience we may quote a paragraph from Durège:[2]

"A complex variable quantity, $z=x+iy$, depends upon two real variables, x and y, which are entirely independent of each other. Hence, for the geometrical representation of a complex quantity, a

[1] In order to meet objections which might be raised to the use of symbolism suggested in this interpretation, it might be well to call attention to the principle which underlies it and makes its use perfectly legitimate, namely, that the operations performed affect only the properties belonging to the symbol and not what the symbol indicates.

[2] See *Theory of Functions*, pp. 13–15.

range of one dimension, a straight line, will no longer suffice, but a region of two dimensions, a plane, will be required for that purpose. The manner of variation of a complex quantity can then be represented by assuming that a point p of the plane is determined by a complex value, $z=x+iy$, in such a way that its rectangular coördinates, in reference to two coördinate axes, assumed to be fixed in the plane, have the values of the real quantities x and y. In the first place, this method of representation includes that of real variables, for when once z becomes real, and therefore $y=0$, the representing point p lies on the axis of x. Next, the coördinates of the point p can vary independently of each other, just as the variables x and y do, so that the point p can change its position in the plane in all directions. Further, one of the two quantities, x and y, can remain constant, while only the other changes its value, in which case the point p will describe a line parallel to the x- or y-axis. Finally and conversely, for every point in the plane the corresponding value of z is fully determined, since by the position of the point p its two rectangular coördinates are given, and therefore also the values of x and y.

"Instead of determining the position of the point p, representing the quantity z by rectangular coördinates x and y, we can accomplish the same by means of polar coördinates, for, by putting

$$r=\cos\phi, \text{ and } y=r\sin\phi,$$

we obtain

$$z=r\,(\cos\phi+i\sin\phi).$$

. . . . Hence we can also say that a complex quantity $r\,(\cos\phi+i\sin\phi)$ represents a straight line, of which the length is equal to r, and which forms an angle ϕ with the principal axis. . . . . From the property of complex quantities, that a combination of two or more of them by means of mathematical operations always leads again to a complex quantity, it follows that, if given complex quantities be represented by points, the result of their combination is capable of being represented by a point."

It is necessary to keep well in mind throughout the discussion the correspondence we have established, the values of our functions, the institutions (G, H, I, J, K, L, M) being expressed in terms of the variable desires (a, b, c, d, e, f). We shall speak as is usual of the variable z, which may stand for any one of our variables or a combination of them forming the complex variable $z=x+iy$. Nor is the method purely artificial, since we are not comparing analogous things, but demonstrating relationships.

A complex variable may describe very different paths in passing from an initial point z_0 to another point z_1. The question arises, "whether the curves described by the function w, starting from w_0, which correspond to those described between z_0 and z_1, must always end in the same point w_1, or whether they cannot also end in different points. Now, in the first place, it is clear that, in the case of *uniform* (one-valued) functions, the final value w_1 must be independent of the path taken; for otherwise the function would be capable of assuming several values for one and the same value of z, which is not possible with uniform functions. This reason, however, does not apply in the case of multiform functions. Such a function has, in fact, several values for the same value of z, and hence the possibility that different paths may also lead to different points or to different values of the function."

To return to the concrete: Suppose our function w is, say, the institution of property in land. Suppose, upon examination, we find the same point reached in New England, where the institution has developed along the lines familiar in history since time past, and, again, out on the Pacific coast, where entirely at variance with their traditions (and, as experience has shown, at variance also with their welfare, and any code of justice that human ingenuity can devise) the same forms have been forced upon the unwilling and deluded Indians by the Land in Severalty bills of Mr. Dawes. Different paths have led to different values of the function, all of which we must know accurately before we may speak with authority concerning the values at given points. To use the symbols again: Let the variable z in $w = \sqrt{z}$ pass from 1–4 by different paths, and let w start with $w = +1$, corresponding to $z = 1$. Let (polar coördinates) $z = r\,(\cos\phi + i \sin\phi)$, then $w = \sqrt{r}\,(\cos \tfrac{1}{2}\phi + i \sin \tfrac{1}{2}\phi)$. Since w is to start with the value $+1$, $r = 1$, and $\phi = 0$; if z describe path from 1–4 and does not inclose the origin, then ϕ arrives at 4 with value 0, while $r = 4 \therefore w = 2$. If z goes once around the origin, then at 4, $\phi = 2\pi$ and $\tfrac{1}{2}\phi = \pi$, while again $r = 4$, hence w acquires the value -2.

The object to be accomplished in introducing the Riemann surface is to convey graphically an idea of the kind of classification which must be made of the different social values (a problem now confronting sociologists) in order that, given any of the human institutions in forms of thought, of personal action, of expression, or of coöperation, we may be able to show corresponding to each, in meaning terms, the variable desires, both in past and present, so that the interdependence

of each and all may be made obvious, and the direction indicated in which any unknown value is to be looked for; just as the discovery of the element argon was predicted from Mendelejeff's chart by the law of the periodic functions of atomic weights.

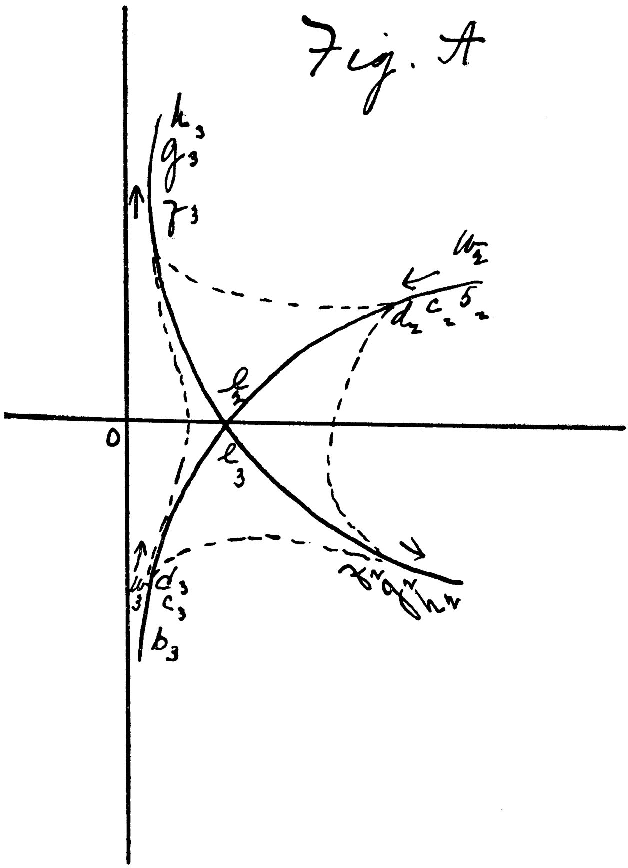
Precisely this task (*i. e.*, the representation) was undertaken by Riemann and the solution found in his conception of the n-sheeted surface.[1]

"Let $w = \sqrt{z}$. To one z in the Argand diagram correspond two values of w, whereas we wish to make two points correspond to two values of w. To these two points the same complex variable z should be attached. Instead of a single z-plane we take two indefinitely thin sheets, one of which lies immediately below the other. To the one value of z correspond two places in these sheets, one vertically below the other. Every place in the upper or lower sheet has one, and only one, of the two values $\sqrt{z}$ or $-\sqrt{z}$ permanently attached to it. If for a given z, $\sqrt{z}$ be attached to the upper sheet, then $-\sqrt{z}$ must be attached to the lower sheet at the point z. When $z = 0$ or ∞, the values of w are equal, and only one place is needed to represent them; hence we regard the sheets as hanging together at 0 and ∞ . . . . If in the Riemann representation the path start from a place in the upper sheet to which the value $\sqrt{z}$ is attached, it must lead to that place in the lower sheet which lies vertically below the initial point. It follows that we must give up the idea of having only one w-value attached to each place of the two sheets, or else make a connection or bridge between the sheets over which every path which goes once around the origin must necessarily pass."[2]

[1] For the formation, etc., see HARKNESS AND MORLEY, *Treatise on the Theory of Functions*, extracts from which are given below.

[2] *Directions for making model of a Riemann surface for a three-valued function:* A difficulty arises, first, because the sheets of the surface interpenetrate, and, in the second place, because frequently at branch-points several sheets, which do not lie one immediately below another, must be supposed to be connected. For the purpose of illustration it is only necessary to be able to follow certain lines in their course through the different sheets of the surface. This may be done as follows: First cut in the three sheets of paper placed one above another, which are to represent the surface, the branch-cuts, and then only at those places where a line is to pass over a branch-cut from one sheet into another join the respective sheets by pasting on strips of paper. Then we can always contrive that, when the line is to return to the first sheet, from which it started, we have the necessary space left for the fastening of the strip of paper by means of which the return passages is effected. By these attached strips union of the separate sheets into one connected surface is accomplished; and it is then no longer necessary to connect the sheets with one another at the branch-points.

The surface in which lie the paths of the different social institutions is composed of n sheets, where n may have an infinite value and (instead of $w = \sqrt[3]{z}$, as in the model) $w = \sqrt[n]{z}$. The area of each sheet is coextensive with the whole of human activity, but the value of the function is uniform (*i. e.*, has only one value corresponding to



one value of the variable) in any one sheet. It will now be necessary to follow with some care the discussion (from Durège) of the cases where either a discontinuity occurs or several function values become equal, the function changing its value in describing a closed line around a *branch-point*. For just as the mathematician applies tests for these points, so the investigator of social forces must proceed to locate the "branch-points" for society, to be able to show when we are

approaching one and ascertain the effect of a passage round, etc. Just what is meant by this will be made clearer after the mathematical exposition.

Consider the function defined by the cubic equation

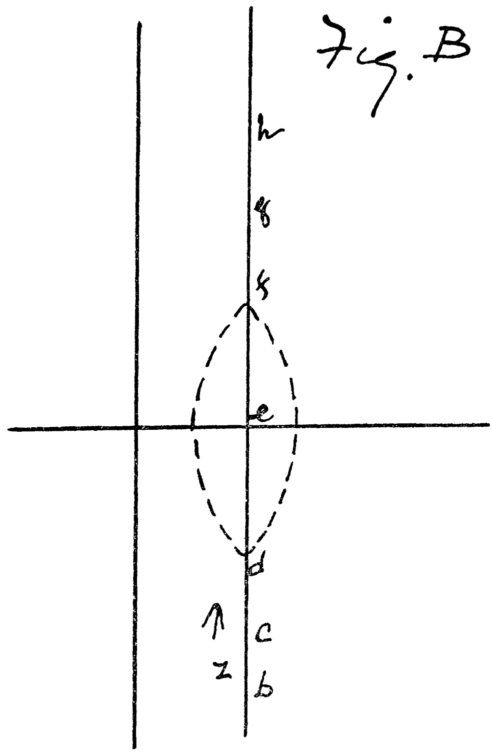
$$w^3 - w + z = 0 \ . \ . \ . \ .$$

"For each value of z, w has in general the three values w_1, w_2, w_3. But the last two of these become equal when $z = \frac{2}{\sqrt{27}}$. At this point we have $w_2 = w_3 = \sqrt{\frac{1}{3}}$. If now we assume that the variable z changes continuously, or that the point representing it describes a line, then each of the three quantities w_1, w_2, w_3 likewise changes continuously, or the three corresponding points describe three separate paths. But when z passes through the point $z = \frac{2}{\sqrt{27}}$, both functions w_2 and w_3 assume the value $\sqrt{\frac{1}{3}}$, hence the two lines w_2 and w_3 meet in the point $\sqrt{\frac{1}{3}}$. At the passage through this, therefore, w_2 can go over into w_3, and w_3 into w_2, without interruption of continuity; indeed, it remains entirely arbitrary on which of the two lines each of the quantities w_2 or w_3 shall continue its course. In this place a branching of the lines described by the quantities w_2 and w_3 takes place, hence Riemann has called those points in the z-plane at which one value of the function can change into another, branch-points.[1]

"In Fig. *A* the three w_1, w_2, w_3 are drawn for the case when z describes a straight line parallel to the y-axis and passing through the branch-point $e = \frac{2}{\sqrt{27}}$. The w-points which correspond to the z-points are denoted by the same letters with attached subscripts 1, 2, 3. Let us follow the path of only one of the quantities, say w_3. This describes the line $b_3\, c_3\, d_3$ and approaches the point $e_3 = e_2 = \frac{1}{\sqrt{3}}$, as z approaches the point $e = \frac{2}{\sqrt{27}}$ along the line $b\, c\, d$. Should z now pass through the point, w_3 could continue its course from $e_2 = e_3 = \sqrt{\frac{1}{3}}$ on either of the two paths $e_3\, f_3\, g_3\, h_3$ or $e_2\, f_2\, g_2\, h_2$, on which one as well

[1] *Definition*—"A point at which either a discontinuity occurs or several function values become equal is called a branch-point when, and only when, the function changes its value in describing a closed line around this and no other similar point." For illustrations of the test applied, see Durège, p. 42.

as the other can be considered the path corresponding to the continuation $e\ f\ g\ h$ of z; the way open to w_3 is, in fact, divided at $e_2 = e_3$ into two branches. When z goes from b to h through the branch-point e, then w_3, starting from b_3, can arrive at h_2 just as well as at h_3; and the same is true of w_3 starting from b_2. In case the path of z leads *through* the branch-point, the final value of the function remains



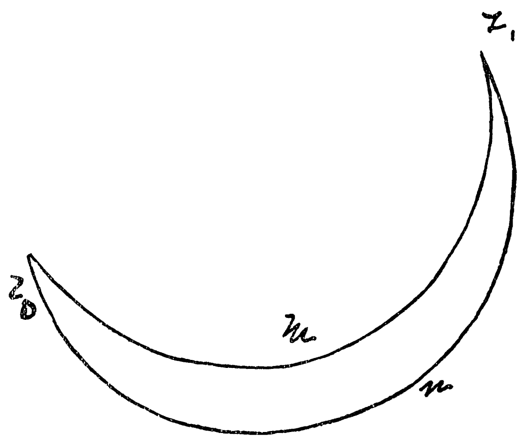
undetermined. If, on the contrary, z describe a path from b to h which does not pass through a branch-point, the final value of the function may differ according to the nature of the path, but it is for each definite path of z *always* completely determined. A similar branching of the function takes place at those points at which w becomes infinite and therefore discontinuous. Thus, for instance, the point $z = 0$ is a branch-point both for the function $w = \frac{1}{\sqrt{z}}$ and for

$w = \sqrt{z}$. Further, in the function determined by the equation

$$(z - b)\,(w - c)^3 = z - a, \text{ or } w = z + \sqrt[3]{\frac{z-a}{z-b}},$$

in which a, b, c denote three complex constants, and therefore three points, $z = a$ is a branch-point, at which all three values of the function become equal to $w = c$. Moreover, at $z = b$ all the values of w become infinite. The three functions suffer here an interruption of continuity, and hence it can remain undetermined on which path each is to continue its course, because, when the function makes a spring, it can just as well spring over to one as to the other continuation of its

Fig. C



path. As a general rule, those points at which w becomes infinite or discontinuous are branch-points. But there are cases in which points are not branch-points, although at them values of the function are either equal or discontinuous. . . . . Accordingly, the branch-points are to be looked for only among those points at which interruption of continuity occurs, or at which several values of the function become equal, but whether such points are actual branch-points must still be expressly determined.

"The preceding considerations have shown that, when a variable z, starting from an arbitrary point z_0, describes a path to another point z_1 which leads through a branch-point of the function w, the latter

acquires different values at z_1, according as it is allowed to proceed on one or the other of its branches. Therefore, in the case of such a path of z, the value of w is undetermined. If, on the contrary, z describe any other path not leading through a branch-point, w acquires at z_1 a definite value, and two paths, both of which lead from z_0 to z_1, assign different values to w only when they inclose a branch-point."

We are as yet pretty fairly baffled in dealing with many phenomena of the life of society. A good illustration, and a case as little understood as any we might choose, is that of *revolutions*, where the functional relations we have been assuming seem at first sight to have been interfered with, and wait to be reëstablished before society regains its equilibrium. We have never yet been able to generalize as to the causes producing, or the changes wrought by, them. All that men have[1] been able to do in times of peril has been to put their fingers on the pulse and give each different guesses as to the direction in which the stream was coursing—little else. Yet there always occurs after such a social upheaval as the French Revolution is generally supposed to have been, in a way precisely similar to the "scattering of the functions at a branch-point," an inexplicable rearrangement of social institutions, and we find them leaving the paths beaten by custom and tradition, and taking entirely new directions. Again "discontinuity occurs." That is, there is left a gap in the path, and, seemingly, no connection remains between the old and the new. There is no need to multiply illustrations. Of the changes in civil government, in industry, in forms of religious freedom, etc., brought about in this way, history can furnish any number. But who has ever been able to predict accurately beforehand the future of any one of the institutions involved?

Now, the task of searching for these branch-points, that we may have warning when we are approaching one, is exactly what the social investigator will one day be expected to perform. But he must have furnished him correct equations of the paths followed by every single one of the human institutions in terms of the variable desires, as they exhibit themselves at any given time and place, the dependence of function upon form of desire being shown to be as close and intimate as though it could be made to appear in some such form as

$$w = \sqrt{(z-a)(z-b)(z-c)(z-d) \ldots .}$$

where $a, b, c, d, \ldots .$ are the branch-points. But remember that the fact that such correlations exist, and the possibility of man's actually

[1] See AMERICAN JOURNAL OF SOCIOLOGY, September, 1897, pp. 161–3.

formulating them in terms of mathematical precision, are two very different things, nor has the latter been here affirmed.

Today, however, the most urgent task of the sociologist is in the present, in learning to understand present conditions. The tedious, but very essential, sort of analysis indicated in this paper is the kind of work before him. Having gained some idea of the interplay of the different forces, some comprehension of the structural interdependence existing between social institutions and human desires, and the functional activity continually at play, he gains also a knowledge of the obstacles which must be removed for his further progress. Before he is able to furnish even skeleton formulæ, in which may be substituted approximate values for the meaning terms of desire, he must have a more complete, a more perfect, classification than has yet been made of the various forms of institutions and desires manifest in actual conditions, the data for which are not yet at hand.

AMY HEWES.